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Few remarks and questions on pseudoradial and related spaces

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Abstract

Some new results concerning pseudoradial and related spaces are presented. Particular emphasis is given to the class of semiradial spaces. In addition, some known facts are collected in order to provide motivations for various questions.

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Dedicated to Luisa M.

The class of pseudoradial spaces has recently received new attention, particularly after 1990 Shapirovskii's theorem on the equivalence of sequential compactness and pseudoradialness in compact spaces (see [24]). A good source of informations for this topic is the 1992 survey article by Nyikos [21].

In this paper we present some new results concerning the various classes of pseudoradial spaces. Particular emphasis is given to the notion of semiradial space. The main purpose of the paper is however to provide motivations for several questions.

Henceforth all spaces are assumed at least Hausdorff. All undefined notions can be found in [14]. $\psi(X)$, $\chi(X)$, $c(X)$, $d(X)$ and $w(X)$ denote respectively the pseudo character, the character, the cellularity, the density and the weight of the space X.

For a sequence we mean any well ordered net.

A sequence $\{x_\alpha\colon \alpha \in \kappa\}$ in a topological space X converges to the point x provided that every neighbourhood of x contains a final segment of it.

Following the notation used in [21], we say that the sequence $\{x_\alpha\colon \alpha \in \kappa\}$ converging to x is thin if $x \notin \overline{\{x_\alpha\colon \alpha \in \beta\}}$ for any $\beta \in \kappa$.

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A subset of a topological space X is said to be chain-closed (thinly chain-closed) (κ-chain-closed) if it contains the limit point of every sequence (thin sequence) (sequence of length not exceeding κ) contained in it. A space X is pseudoradial if all its chain-closed subsets are closed.

A space X is almost radial if all its thinly chain-closed subsets are closed.

A space X is radial if whenever $x \in \overline{A}$ there exists a sequence in A which converges to x.

A space X is R-monolithic if every $|A|$-chain-closed subset of $\overline{A}$ is closed for any $A \subset X$.

Proposition 1. *Every radial space and every R-monolithic space is almost radial and every almost radial space is pseudoradial. Moreover, every Fréchet–Urysohn space is radial and every sequential space is R-monolithic.*

The class of pseudoradial spaces has a very bad behaviour with respect to the product operation. To realize this, look at the following example. It shows that even the product of two very good radial spaces, one compact metric and the other a Lindelöf space with only one isolated point, may fail to be pseudoradial.

Let $X = \omega_1 \cup \{p\}$ be the one Lindelöfization of the discrete space ω_1 and let I be the unit segment with the euclidian topology. Fix a one to one mapping $f : \omega_1 \to I$ and let $A = \{(\alpha, f(\alpha))\colon \alpha \in \omega_1\} \subset X \times I$. Observe first that no sequence contained in A can converge outside A. Indeed, if $S \subset A$ is a sequence converging to a point outside A, then $\pi_X(S)$ must converge to p and hence $|S| = \omega_1$. On the other hand, since I is first countable, $\pi_I(S)$ cannot converge to any point. But A is not closed. If $x \in I$ is a complete accumulation point of $f(\omega_1)$, then $(p, x) \in \overline{A} \setminus A$.

The example just discussed consists of a product of a compact space with a Lindëlof one. This leaves a certain hope to find some positive result on the productivity of pseudoradialness in the class of compact spaces. The next paragraphs will clarify this point.

First of all let us mention the improvement obtained by Juhász and Szentmiklóssy of Shapirovskii's theorem.

Theorem 1 [19]. (a) *If $\mathfrak{c} \leqslant \omega_2$, then every compact sequentially compact space is pseudoradial.*

(b) *It is consistent with* ZFC + ($\mathfrak{c} = \omega_3$) *the existence of a compact sequentially compact space which is not pseudoradial.*

An important immediate consequence of part (a) is:

Theorem 2. *If $\mathfrak{c} \leqslant \omega_2$, then the product of countably many compact pseudoradial spaces is pseudoradial.*

It is not known whether in Theorem 2 the extra set-theoretic assumption can be removed, even in a product with only two factors. In fact this problem is to be considered fundamental for a better understanding of the structure of pseudoradial spaces.

A consequence of Theorem 1(a) of purely consistent nature is:

Theorem 2′. *The sentence "the product of ω_1 compact pseudoradial spaces is always pseudoradial" is independent of* ZFC.

Proof. The assertion is false assuming $\mathfrak{c} = \omega_1$, for instance in this case the Cantor cube 2^{ω_1} is not sequentially compact (see [26]). On the other hand, the assertion is true in any model of $\mathfrak{c} = \omega_2$ in which sequential compactness is ω_1 productive. For instance it suffices to assume $\mathfrak{p} = \mathfrak{c} = \omega_2$ (see [26]). □

The remark made after Theorem 2 naturally leads us to look at what we can actually prove for the product in ZFC.

In order to formulate the best result of this sort, we need to consider a new class of pseudoradial spaces.

Recall that a subset A of a topological space X is κ-closed if $\overline{B} \subset A$ for any $B \subset A$ with $|B| \leqslant \kappa$.

A space X is said to be semiradial provided that all its κ-chain-closed subsets are κ-closed.

The class of semiradial spaces contains both radial and R-monolithic spaces and is contained in the class of almost radial spaces. There exists a compact semiradial space which is neither radial nor R-monolithic. For this, it suffices to consider the topological sum of a compact sequential non-Fréchet–Urysohn space and the space $\delta(N)$ of [23, Example 4.3], which is a compact separable radial nonsequential space. In [11] the question about the existence of a compact almost radial nonsemiradial space was implicitly left open. Below we describe the construction of such a space. The author is indebted to A. Dow for calling his attention to it.

Following [13], we call a tower a well ordered, by reverse almost inclusion, family of infinite subsets of ω with no infinite pseudo-intersection. In other words, the family $\{B_\alpha\colon\ \alpha \in \gamma\}$ of infinite subsets of ω is a tower provided that:

(i) if $\alpha < \beta$, then $|B_\alpha \setminus B_\beta| = \omega$ and $|B_\beta \setminus B_\alpha| < \omega$;

(ii) if A is an infinite subset of ω, then there exists some α such that $|A \setminus B_\alpha| = \omega$.

The smallest cardinality of a tower is denoted by $\mathfrak{t}$. Now let $\{A_\alpha\colon\ \alpha \in \mathfrak{t}\}$ be a family of subsets of ω such that $\{\omega \setminus A_\alpha\colon\ \alpha \in \mathfrak{t}\}$ is a tower. Furthermore put $A_\mathfrak{t} = \omega$. Define a topology on the set $X = \omega \cup \{x_\alpha\colon\ \alpha \leqslant \mathfrak{t}\}$ by declaring each point of ω isolated and taking as a local base at x_α the sets $\{x_\gamma\colon\ \beta < \gamma \leqslant \alpha\} \cup (A_\alpha \setminus (A_\beta \cup F))$, where F is a finite subset of ω and $\beta < \alpha$. It is easy to see that the space X is compact. We check first that X is almost radial. Let A be a non closed subset of X and let γ be the smallest ordinal such that $x_\gamma \in \overline{A} \setminus A$. If $x_\gamma \in \overline{A \setminus \omega}$, then since the subspace $X \setminus \omega$ is homeomorphic to the set $\mathfrak{t} + 1$ equipped with the order topology, there exists a thin sequence $S \subset A \setminus \omega$ converging to x_γ. On the other hand, if $x_\gamma \notin \overline{A \setminus \omega}$, then there exists some $\beta < \gamma$ such that $x_\alpha \notin A$ for $\beta < \alpha < \gamma$. By the minimality of γ, we have $x_\alpha \notin \overline{A} \setminus A$ for $\alpha < \gamma$ and therefore we may conclude that $x_\alpha \notin \overline{A}$ for $\beta < \alpha < \gamma$. This means that the set $\{x_\gamma\} \cup A \setminus A_\beta$ is closed in X. Since $\{x_\gamma\} \cup A \setminus A_\beta$ is countable and has x_γ as the only non-isolated point, there clearly exists a sequence in $A \setminus A_\beta$ converging to x_γ. Thus X is almost radial. To check that X is not semiradial, notice first that $X \setminus \{x_\mathfrak{t}\}$ is sequentially compact. Indeed, let $S \subset X \setminus \{x_\mathfrak{t}\}$ be a countable set. If $S \setminus \omega$ is infinite, then since $\mathfrak{t}$ is a regular uncountable cardinal, S has a convergent subsequence in $X \setminus \{x_\mathfrak{t}\}$. If $S \subset \omega$, then by property (ii) there exists some $\alpha \in \mathfrak{t}$ such that $|S \cap A_\alpha| = \omega$ and we see that the set $S \cap A_\alpha$ is a sequence converging to x_α. To finish, observe that $X \setminus \{x_\mathfrak{t}\}$ is not ω-closed, but no countable subsequence of it can converge to $x_\mathfrak{t}$.

The following is at moment the wider result concerning the productivity of pseudoradialness in ZFC.

Theorem 3 [11]. *The product of a compact semiradial space and a compact pseudoradial space is pseudoradial.*

As an immediate consequence of this theorem, we get some other results which were known before.

Corollary 1 [15]. *The product of a compact pseudoradial space and a compact radial space is pseudoradial.*

Corollary 2 [7]. *The product of a compact pseudoradial space and a compact R-monolithic space is pseudoradial.*

Moreover, we have:

Corollary 3. *The product of finitely many compact semiradial spaces is pseudoradial.*

The last corollary suggests:

Question 1 (ZFC). Is the product of countably many compact semiradial (or at least radial) spaces pseudoradial?

For countable products the situation is much better in the class of R-monolithic spaces.

Theorem 4 [8]. *The class of R-monolithic compact spaces is countably productive.*

The argument used in the proof of Theorem 4 could be applied to get a positive answer to Question 1 if we were able to show that the product of two compact semiradial spaces is also semiradial.

Recall that the cardinal $\mathfrak{p}$ is the smallest cardinality of a family $\mathcal{A} \subset [\omega]^\omega$ closed under finite intersection such that there exists no set $B \in [\omega]^\omega$ satisfying $|B \setminus A| < \omega$ for every $A \in \mathcal{A}$.

In 1989 Nyikos published a condition for the almost radialness of certain spaces involving the cardinal $\mathfrak{p}$, precisely:

Theorem 5 [20]. *If $\mathfrak{p} > \omega_1$ then every countably compact regular space with character not exceeding ω_1 is almost radial.*

This result can be formally improved.

Although the proof of Theorem 5′ is like the one of Theorem 5, for completeness we give the details of it.

Theorem 5′. *If $\mathfrak{p} > \omega_1$, then every countably compact regular space with character not exceeding ω_1 is semiradial.*

Proof. Let X be a countably compact regular space with character not exceeding ω_1 and let A be a nonclosed subset of X. If A is not ω-closed, then fix a countable set $B \subset A$ for which $\overline{B} \setminus A \neq \emptyset$. Taking a point $x \in \overline{B} \setminus A$, a standard application of $\mathfrak{p} > \omega_1$ provides a subsequence of B converging to x. Assume now that A is ω-closed, let $x \in \overline{A} \setminus A$ and fix a local base $\{U_\alpha\colon \alpha \in \omega_1\}$. Since A is countably compact the set $\bigcap_{\beta\in\alpha} \overline{U_\beta} \cap A$ is not empty and we can choose a point x_α in it. By the regularity of X the sequence $\{x_\alpha\colon \alpha \in \omega_1\}$ converges to x. This shows that X is semiradial because in X a nonclosed set is either not ω-closed or not ω_1-closed. □

In [20] Nyikos also asked whether Theorem 5 could be reversed. In view of Theorem 5′, this question may be reformulated in an easier one.

Recall that a space X is weakly monolithic if $\psi(\overline{A}) \leqslant |A|$ for every $A \subset X$.

Proposition 2 [3]. *Every compact weakly monolithic space is R-monolithic.*

To prove our next result we need the following strengthening of the countable productivity of the class of R-monolithic spaces established in Theorem 4.

Theorem 6 [9, Theorem 4]. *Let $\{X_\alpha\colon \alpha \in \kappa\}$ be a family of R-monolithic compact spaces. If $\lambda \geqslant \kappa$, then every $< \kappa$-closed λ-chain-closed subset of $\prod_{\alpha\in\kappa} X_\alpha$ is λ-closed.*

Theorem 7. *If $\mathfrak{p} > \omega_1$, then the product of ω_1 compact weakly monolithic spaces is semiradial.*

Proof. Let $X = \prod_{\alpha\in\omega_1} X_\alpha$ be as in the hypotheses and let A be a nonclosed subset of X. Denote by κ the smallest cardinal such that A is not κ-closed. If $\kappa > \omega$, then by Theorem 6 we see that A is not κ-chain-closed. If $\kappa = \omega$ then there exists a countable set $B \subset A$ satisfying $\overline{B} \setminus A \neq \emptyset$. The set $\overline{B}$ is contained in the product of ω_1 closed separable, and hence first countable by the weak monolithicity of the factors of X, spaces. It then follows that $\chi(\overline{B}) \leqslant \omega_1$. Now if $x \in \overline{B} \setminus A$, then a standard application of $\mathfrak{p} > \omega_1$ allows us to find a subsequence of B converging to x. Thus A is not ω-chain-closed and consequently X is semiradial. □

The class of weakly monolithic spaces contains all linearly orderable spaces (LOTs).

Corollary 4. *If $\mathfrak{p} > \omega_1$, then the product of ω_1 compact LOTs is semiradial.*

Proposition 2 and Theorem 4 clearly suggest:

Question 2. Does $\mathfrak{p} > \omega_1$ imply that the product of ω_1 compact R-monolithic spaces is semiradial?

As another application of the number $\mathfrak{p}$, more precisely of the axiom $\mathfrak{p} = \mathfrak{c}$, we have the following result of Juhász and Szentmiklóssy [19, Theorem 6]: if $\mathfrak{p} = \mathfrak{c}$, then every compact sequentially compact space of countable tightness is pseudoradial. Our next result shows that in the previous theorem compactness is actually not necessary.

Theorem 8 ($\mathfrak{p} = \mathfrak{c}$). *Every sequentially compact regular space of countable tightness is pseudoradial.*

Proof. Let X be a sequentially compact regular space with countable tightness and fix a nonclosed set $A \subset X$. If A is not countably compact, then there exists a countable infinite set $B \subset A$ without accumulation points in A. Since X is sequentially compact, the set B has a convergent subsequence and the limit point of it is clearly outside A. Suppose now that A is countably compact and select a countable set $B \subset A$ such that $\overline{B} \setminus A \neq \emptyset$. Let $x \in \overline{B} \setminus A$ and observe that the regularity of X implies $\chi(x, \overline{B}) \leqslant \mathfrak{c}$. If $\chi(x, \overline{B}) < \mathfrak{c}$, then $\mathfrak{p} = \mathfrak{c}$ implies the existence of a subsequence of B converging to x. If $\chi(x, \overline{B}) = \mathfrak{c}$, then fix a local base $\{U_\alpha\colon \alpha < \mathfrak{c}\}$ of x in the subspace $\overline{B}$. By $\mathfrak{p} = \mathfrak{c}$, for any $\alpha < \mathfrak{c}$ there exists an infinite set $B_\alpha \subset B$ such that $B_\alpha \setminus U_\beta$ is finite for each $\beta \leqslant \alpha$. For every α the set B_α has an accumulation point $x_\alpha \in A$ and, by construction, $x_\alpha \in \overline{U_\beta}$ for each $\beta \leqslant \alpha$. The last assertion together with the regularity of X imply that the sequence $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ converges to x and the proof is complete. □

In a similar manner we can obtain in ZFC a result already stated by Nyikos in [20].

Theorem 9. *Every sequentially compact regular space with character not exceeding ω_1 is pseudoradial.*

An interesting question arises in considering the cardinality of a pseudoradial space. Well-known facts concerning the cardinality properties are collected in the following:

Proposition 3 (see [1] and [2]).

- *If X is compact radial, then $|X| \leqslant 2^{c(X)}$.*
- *If X is compact sequential, then $|X| \leqslant 2^{c(X)}$.*
- *If X is compact pseudoradial, then $w(X) \leqslant 2^{c(X)}$.*

The content of this proposition leads clearly to wonder whether it is true that $|X| \leqslant 2^{c(X)}$ for any compact pseudoradial space X.

At least consistently this is not the case. In fact, it was observed in [4] that there is a model of ZFC where the Cantor cube 2^{ω_1} is pseudoradial and $2^{\omega_1} > \mathfrak{c}$.

It remains unclear what happen in ZFC. Precisely we can ask:

Question 3. Does there exist in ZFC a compact pseudoradial space X for which $|X| > 2^{c(X)}$?

A space with countable tightness cannot answer Question 3, in fact by Balogh's theorem [6] every compact space with countable tightness is sequential assuming PFA. In other models, however, we can have compact pseudoradial nonsequential spaces with countable tightness, for instance the one point compactification of the Ostaszewski's space [22]. Notice that the cardinality of this space is small, namely $\omega_1 = \mathfrak{c}$ and it seems very reasonable to ask the following:

Question 4. Is it true that $|X| \leqslant 2^{c(X)}$ for every compact pseudoradial space with countable tightness?

A very partial positive result concerning the cardinality of a space is:

Theorem 10 [4]. *If X is an R-monolithic compact homogeneous space, then $|X| \leqslant 2^{c(X)}$.*

The homogeneity of the space is however a very strong requirement and it would be nice to have a positive answer to:

Question 5. Is it true that $|X| \leqslant 2^{c(X)}$ for any compact R-monolithic (or at least monolithic) space X?

The reasonability of the above question is supported also by the following:

Theorem 11 [5]. *If* MA + $\neg$CH *holds, then the cardinality of a compact monolithic space with countable cellularity does not exceed the continuum.*

It follows directly from the definition that:

Proposition 4. *If X is semiradial, then $|X| \leqslant 2^{d(X)}$.*

Following Juhász and Nyikos [18], a space X is said to be tame provided that $|\overline{A}| \leqslant 2^{|A|}$ for every $A \subset X$. Proposition 4 then says that a semiradial space is tame.

In [18] the property of omitting cardinals in tame spaces was studied and therefore we may wonder whether semiradial spaces can play any role in this area.

Recall that a space X omits the cardinal κ if $|X| > \kappa$ and no closed subset of X has cardinality κ. In [18, Lemma 2] the following is given:

Proposition 5. *If X is a generalized orderable space and $\kappa^{<\kappa} = \kappa$, then X does not omit κ.*

Every generalized orderable space is radial and we have:

Theorem 12. *If $\kappa^{<\kappa} = \kappa$, then no pseudoradial tame space omits κ.*

Proof. Observe first that $\kappa^{<\kappa} = \kappa$ implies that κ is regular. Let X be a pseudoradial tame space. Assume first that there exists in X a non trivial sequence $S = \{x_\alpha \colon \alpha \in \kappa\}$ converging to some x. For every α we have $|\overline{\{x_\beta \colon \beta \in \alpha\}}| \leqslant 2^{|\alpha|} \leqslant \kappa$, by tameness and $\kappa^{<\kappa} = \kappa$. Thus the closed set $\overline{S} = \{x\} \cup (\bigcup_{\alpha\in\kappa} \overline{\{x_\beta \colon \beta \in \alpha\}})$ has cardinality κ. Now, suppose that X has no convergent sequence of length κ and let A be a set of cardinality κ. Let $A_0 = A$ and A_α be the set of all limit points of the convergent sequences of length less than κ which are contained in $\bigcup_{\beta\in\alpha} A_\beta$. Since $\kappa^{<\kappa} = \kappa$, it follows that $|A_\alpha| = \kappa$ for each α and so also $|\bigcup_{\alpha\in\kappa} A_\alpha| = \kappa$. We claim that $\overline{A} = \bigcup_{\alpha\in\kappa} A_\alpha$. If not, the latter set is not closed and there exists a sequence $S \subset \bigcup_{\alpha\in\kappa} A_\alpha$ converging to a point outside that set and satisfying $|S| \leqslant \kappa$. In fact we must have $|S| < \kappa$ and therefore $S \subset A_\alpha$ for some $\alpha \in \kappa$. This shows that the limit point of S is actually in $\bigcup_{\alpha\in\kappa} A_\alpha$ and we reach a contradiction. So $\overline{A}$ is a set of cardinality κ and we are done. □

In view of Proposition 4 we have:

Corollary 5. *If $\kappa^{<\kappa} = \kappa$, then no semiradial space omits κ.*

At least consistently (see [4]) a pseudoradial space may fail to be tame. However, it is not clear whether Theorem 12 can be extended to the whole class of pseudoradial spaces, or at least to the whole class of almost radial spaces. Clearly the latter case has more chances to be proved true via a strengthening of Proposition 4. This leads us to the following:

Question 6. Is $|X| \leqslant 2^{d(X)}$ for every (compact) almost radial space X?

Recall that $C_p(X)$ denotes the space of all real valued continuous functions defined on the Tychonoff space X equipped with the topology of pointwise convergence.

A well known result (see, e.g., [16]) says that $C_p(X)$ is Fréchet–Urysohn if and only if it is sequential.

Regarding the higher convergence property of a function space, it was shown in [17] that $C_p(X)$ is radial if and only if it is Fréchet–Urysohn, but radiality and pseudoradiality do not in general coincide for function spaces. In particular we have the following:

Proposition 6 [17]. *If ξ is an ordinal, then:*

(a) *$C_p(\xi)$ is Fréchet–Urysohn if and only if* $\mathrm{cf}(\xi) \leqslant \omega$;

(b) *$C_p(\xi)$ is pseudoradial if and only if ξ is regular and $\lambda^\omega < \xi$ for every $\lambda < \xi$.*

Successively, it was observed in [12] that for any ordinal ξ the space $C_p(\xi)$ is pseudoradial if and only if it is almost radial. Thus even radiality and almost radiality do not, in general, coincide for function spaces.

It remains unclear what happens with the more special notion of semiradial and R-monolithic space. In particular, taking into account that a R-monolithic space is somehow very near a sequential space, one can wonder whether every R-monolithic space of the form $C_p(X)$ must necessarily be sequential.

The answer is in the negative, at least assuming the existence of a strongly inaccessible cardinal, that is an uncountable regular cardinal κ satisfying $2^\lambda < \kappa$ for each $\lambda < \kappa$.

Theorem 13. *If ξ is a strongly inaccessible cardinal, then $C_p(\xi)$ is R-monolithic.*

Proof. Observe first that, by Proposition 6(b), if ξ is strongly inaccessible, then $C_p(\xi)$ is pseudoradial. In [12, Lemma 3.14], it was remarked that for any regular cardinal κ the only nontrivial convergent sequences in $C_p(\kappa)$ have either length ω or length κ. This clearly implies that every closed subspace of $C_p(\xi)$ of cardinality less than ξ is actually sequential. To verify the R-monolithicity of $C_p(\xi)$, it suffices to look at the subsets A of $C_p(\xi)$ having cardinality less than ξ. For such a set we have $|\overline{A}| \leqslant 2^{2^{|A|}} < \xi$ and hence $\overline{A}$ is sequential.

Strengthening Proposition 6(b), we may look for a positive answer to the following:

Question 7. Is it true that $C_p(\xi)$ is R-monolithic if and only if ξ is regular and satisfies $\lambda^\omega < \xi$ for each $\lambda < \xi$?

We close the paper looking at a recent class of spaces which has strong connections with the class of pseudoradial spaces.

Following Simon [25] we say that the space X has the property of weak approximation by points (briefly WAP) if for every nonclosed set $A \subset X$ there exists a set $B \subset A$ such that $|\overline{B} \setminus A| = 1$.

Proposition 7 [10]. (a) *Every semiradial space is a WAP space.*

(b) *Every compact WAP space is pseudoradial.*

Compactness in part (b) is necessary, for instance if $p \in \beta(\omega) \setminus \omega$, then $\omega \cup \{p\}$ is a Lindelöf WAP space which is not pseudoradial.

In [25] Simon showed that the product of a Lindelöf WAP space with the unit interval can fail to be a WAP space.

In the compact case it turns out that WAP spaces behave much better under the product operation.

Theorem 14 [10]. *The product of a compact WAP space with a compact semiradial space is a WAP space.*

In particular, then:

Corollary 6. *The product of a compact WAP space with the unit interval is a WAP space.*

Clearly, if we could show that the second part of Proposition 7 is reversible, then Theorem 14 would be equivalent to Theorem 3. Thus we ask:

Question 8. Does there exist a compact pseudoradial non-WAP space?

Or in addition:

Question 9. Is the class of compact WAP spaces finitely productive?

In [21, Problem 5.10], Nyikos asked whether there is a model of ZFC in which every non-isolated point of a compact space is a limit of a nontrivial thin sequence. Arguing as in the proof of Proposition 7(b), we see that in such a model every compact WAP space is almost radial. A weak version of Nyikos' problem is, then:

Question 10. Is it consistent with ZFC that every compact WAP space is almost radial?

Note added in proof

P. Simon and G. Tironi have shown that CH implies the existence of two compact semiradial spaces whose product is not semiradial.

References

[1] A.V. Arhangel'skii, Suslin number and cardinality. Character of points in sequential compact Hausdorff spaces, Soviet Math. Dokl. 11 (1970).

[2] A.V. Arhangel'skii, Some properties of radial spaces, Math. Notes Russ. Acad. Sci. 27 (1980) 50–54.

[3] A.V. Arhangel'skii and A. Bella, On the product of biradial compact spaces, Topology Appl. 45 (1992) 157–162.

[4] A.V. Arhangel'skii and A. Bella, On the cardinality of a pseudoradial compact space, Boll. Un. Mat. It. 7a (1993) 237–241.

[5] A.V. Arhangel'skii and B.E. Shapirovskii, The structure of monolithic spaces, Moscow Univ. Math. Bull. (1987).

[6] Z. Balog, On compact Hausdorff spaces of countable tightness, Proc. Amer. Math. Soc. 105 (1989) 755–764.

[7] A. Bella, More on the product of pseudoradial spaces, Comment. Math. Univ. Carolin. 32 (1991) 125–128.

[8] A. Bella, Countable productivity of a class of pseudoradial spaces, Proc. Amer. Math. Soc. 119 (1993) 637–640.

[9] A. Bella, Cleavability, pseudoradial and R-monolithic spaces, Topology Appl. 53 (1993) 229–237.

[10] A. Bella, On spaces with the property of weak approximation by points, Comment. Math. Univ. Carolin. 35 (1994) 357–360.
[11] A. Bella and J. Gerlits, On a condition for the pseudoradiality of a product, Comment. Math. Univ. Carolin. 33 (1992) 311–313.
[12] G. Dimov and G. Tironi, Some remarks on almost radiality in function spaces, Acta Univ. Carolin. 28 (1987) 49–58.
[13] E.K. van Douwen, The Integers and Topology, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (Elsevier, Amsterdam, 1984) 111–167.
[14] R. Engelking, General Topology (Heldermann-Verlag, Berlin, 1989).
[15] Z. Frolik and G. Tironi, Products of chain-net spaces, Riv. Mat. Pura Appl. Udine 5 (1989) 7–11.
[16] J. Gerlits and Z.S. Nagy, Some properties of $C(X)$ I, Topology Appl. 14 (1982) 151–161.
[17] J. Gerlits, Z.S. Nagy and Z. Szentmiklóssy, Some convergence properties in function spaces, in: Z. Frolik, ed., Proceedings of the 6th Prague Topological Symposium (Heldermann-Verlag, Berlin, 1988) 211–222 .
[18] I. Juhász and P.J. Nyikos, Omitting cardinals in tame spaces, Colloq. Math. 57 (1989) 193–202.
[19] I. Juhász and Z. Szentmiklossy, Sequential compactness versus pseudoradiality in compact spaces, Topology Appl. 50 (1993) 47–53.
[20] P.J. Nyikos, Sequential properties of 2^{ω_1} under various axioms, in: Baku International Topological Conference Proceedings (1989) 314–322.
[21] P.J. Nyikos, Convergence in topology, in: M. Hušek and J. van Mill, eds., Recent Progress in General Topology (Elsevier Science Publishers B.V., Amsterdam, 1992) 537–570.
[22] A. Ostaszewski, On countably compact, perfectly normal spaces, J. London Math. Soc. 14 (1976) 505–516.
[23] K. Malliha Devi, P.R. Meyer and M. Rajagopalan, When does countable compactness imply sequential compactness?, Gen. Topology Appl. 6 (1976) 279–289.
[24] B.E. Shapirovskii, Orderings and cardinal invariants in compacta, in: Abstracts, Topological Conferences of Varna (1990).
[25] P. Simon, On accumulation points, Cahiers Topologie Géom. Différentielle Catégoriques XXXV (4) (1994) 321–327.
[26] J.E. Vaughan, Countably compact and sequentially compact spaces, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (Elsevier Science Publishers B.V., Amsterdam, 1984).